

GRADIENT BOUNDS FOR P-HARMONIC SYSTEMS WITH VANISHING NEUMANN DATA IN A CONVEX DOMAIN

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ABSTRACT. Let $\tilde{\Omega}$ be a bounded convex domain in Euclidean n space, $\hat{x} \in \partial\tilde{\Omega}$, and $r > 0$. Let $\tilde{u} = (\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^N)$ be a weak solution to $\nabla \cdot (|\nabla \tilde{u}|^{p-2} \nabla \tilde{u}) = 0$ in $\tilde{\Omega} \cap B(\hat{x}, 4r)$ with $|\nabla \tilde{u}|^{p-2} \tilde{u}_\nu = 0$ on $\partial\tilde{\Omega} \cap B(\hat{x}, 4r)$.

We show that sub solution type arguments for certain uniformly elliptic systems can be used to deduce that $|\nabla \tilde{u}|$ is bounded in $\tilde{\Omega} \cap B(\hat{x}, r)$ with constants depending only on n, p, N , and $\frac{r^n}{|\tilde{\Omega} \cap B(\hat{x}, r)|}$. Our argument replaces an argument based on level sets in recent important work of [CM], [CM1], [GS], [M], [M1], involving similar problems.

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1. INTRODUCTION

Let $x = (x_1, \dots, x_n)$ denote points in Euclidean n space, \mathbb{R}^n , and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^n . let $|x| = \langle x, x \rangle^{1/2}$ denote the norm of x and set $B(x, \rho) = \{y : |y - x| < \rho\}$ when $\rho > 0$. Given an open set $O \subset \mathbb{R}^n$, let $C_0^\infty(O)$ denote infinitely differentiable functions with compact support in O . If $1 \leq q \leq \infty$, let $W^{1,q}(O)$ denote the Sobolev space of functions g with distributional derivatives g_{x_i} , $1 \leq i \leq n$, and norm,

$$\|g\|_{W^{1,q}(O)} = \|g\|_{L^q(O)} + \| |\nabla g| \|_{L^q(O)} < \infty.$$

Here ∇g denotes the gradient of g and $\|\cdot\|_{L^q(O)}$ is the usual Lebesgue q norm relative to O . If $E, F \subset \mathbb{R}^n$ let $\hat{H}(E, F)$ denote the Hausdorff distance between the sets E, F , and let $|E|$ denote the Lebesgue n measure of E whenever E is measurable. Throughout this paper we assume that $\tilde{\Omega} \subset \mathbb{R}^n$ is a bounded convex domain and for given $\hat{x} \in \tilde{\Omega}$, $1 < p < \infty$, $r > 0$, that $\tilde{u} = (\tilde{u}^1, \dots, \tilde{u}^N) : \tilde{\Omega} \rightarrow \mathbb{R}^N$ is a weak solution to the p Laplace systems equation in $\tilde{\Omega} \cap B(\hat{x}, 4r)$ with vanishing Neumann data on $\partial\tilde{\Omega} \cap B(\hat{x}, 4r)$.

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That is, $\tilde{u}^k \in W^{1,p}(\tilde{\Omega} \cap B(\hat{x}, 4r))$, $1 \leq k \leq N$, and if $\phi = (\phi^1, \dots, \phi^N)$,

$$(1.0) \quad \int_{\tilde{\Omega} \cap B(\hat{x}, 4r)} |\nabla \tilde{u}|^{p-2} \langle \nabla \tilde{u}, \nabla \phi \rangle dx = 0$$

whenever $\phi^i \in W^{1,p}(\tilde{\Omega} \cap B(\hat{x}, 4r))$ and ϕ^i , $1 \leq i \leq N$, vanishes outside a set whose closure is a compact subset of $B(\hat{x}, 4r)$. In the above display, $\nabla \tilde{u}$ is the nN tuple, $(\nabla \tilde{u}^1, \dots, \nabla \tilde{u}^N)$ while the inner product is relative to \mathbb{R}^{nN} . In this note we show that

Theorem 1.1. *Let $\tilde{u}, \hat{x}, r, \tilde{\Omega}, p$ be as above. There exists C depending only on $n, p, N, r^n/|\Omega \cap B(\hat{x}, r)|$, such that for every $y \in \tilde{\Omega} \cap B(\hat{x}, r)$,*

$$|\nabla \tilde{u}|^p(y) \leq C r^{-n} \int_{\tilde{\Omega} \cap B(\hat{x}, 4r)} |\nabla \tilde{u}|^p dx$$

We note that in [CM], [CM1], the authors studied weak solutions to quasi-linear elliptic equations and systems of the form $\nabla \cdot (a(|\nabla \tilde{u}|) \nabla \tilde{u}) = f(x)$ in a convex domain $\tilde{\Omega}$ under both Dirichlet and Neumann boundary conditions: $\tilde{u} \equiv 0, \tilde{u}_\nu \equiv 0$ respectively on $\partial \tilde{\Omega}$. It is not obvious to us that their nearly endpoint global type results imply Theorem 1.1. Our proof though uses some of the same arguments as in these papers, including fundamental use of an inequality in [G] for convex domains. Also as in these papers, we first prove Theorem 1.1 for a weak solution to a related PDE in a smooth convex domain and then use a limiting argument to get Theorem 1.1. However the above authors use the level sets of $|\nabla u|$ and perform some rather involved calculations on these sets in order to obtain their results. In contrast we use well known sub solution type arguments for $|\nabla \tilde{u}|$ (i.e, Moser iteration) to get Theorem 1.1.

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2. PROOF OF THEOREM 1.1

To begin the proof of Theorem 1.1 let c be a positive constant, not necessarily the same at each occurrence, which unless otherwise stated may only depend on n, p, N . We note that the p Laplace equation is invariant under dilations, translations, and scaling. Thus it suffices to prove Theorem 1.1 when

$$(2.0) \quad r = 1, \hat{x} = 0, \text{ and } \int_{\tilde{\Omega} \cap B(0, 4)} |\nabla \tilde{u}|^p dx = 1,$$

as we can then transfer back to the general case using the transformation $y = \hat{x} + rx$, after multiplying \tilde{u} by an appropriate constant. We continue under assumption (2.0). We may also assume that $\tilde{\Omega} \cap \partial B(0, t) \neq \emptyset$ whenever $t < 4$ since otherwise it follows from (1.0) with $\phi =$ a suitable extension of

\tilde{u} to $B(0, 4)$ that $\tilde{u} \equiv \text{constant}$. Let $\rho, 3 < \rho < 4$, be such that

$$(2.1) \quad \int_{\tilde{\Omega} \cap \partial B(0, \rho)} |\nabla \tilde{u}|^p d\sigma \leq c$$

where σ denotes Hausdorff $n - 1$ measure = surface area. Existence of ρ, c follows from (2.0), writing the integral in polar coordinates. and the usual weak type estimates. Moreover, (2.1) holds for ρ belonging to a set of positive measure in $[3, 4]$. Let $\eta(x) = \gamma(|x|)$ where $\gamma \equiv -1$ in $[0, \rho - \delta]$, linear in $[\rho - \delta, \rho]$ and $\gamma(t) = 0$ for all $t \geq \rho$. Using this η as a test function in (1.0) and letting $\delta \rightarrow 0$, we deduce from the Lebesgue Differentiation theorem, that we may assume

$$(2.2) \quad \int_{\tilde{\Omega} \cap \partial B(0, \rho)} |\nabla \tilde{u}|^{p-2} \tilde{u}_\nu d\sigma = 0$$

where $\nu(x)$ denotes the outer unit normal to $\tilde{\Omega} \cap B(0, \rho)$ at $x \in \tilde{\Omega} \cap \partial B(0, \rho)$ and $\tilde{u}_\nu = (\tilde{u}_\nu^1, \dots, \tilde{u}_\nu^N)$ with $\tilde{u}_\nu^k(x) = \langle \nabla \tilde{u}^k(x), \nu(x) \rangle, 1 \leq k \leq N$. Next given $\epsilon > 0, 0 < \epsilon < 1/2$, let $\Omega = \Omega(\epsilon)$ be a convex domain with

$$(2.3) \quad \tilde{\Omega} \subset \Omega, \quad \hat{H}(\partial\Omega, \partial\tilde{\Omega}) < \epsilon, \quad \text{and} \quad \partial\Omega \in C^\infty.$$

Let ρ be as in (2.2) and let $f = (f^1, \dots, f^N)$ be defined by $f^k(x) = 0, 1 \leq k \leq N$, when $x \in (\Omega \setminus \tilde{\Omega}) \cap \partial B(0, \rho)$ while $f = |\nabla \tilde{u}|^{p-2} \tilde{u}_\nu$, otherwise on $\Omega \cap \partial B(0, \rho)$. Note from (2.2) that f satisfies the compatibility condition :

$$(2.4) \quad \int_{\Omega \cap \partial B(0, \rho)} f^k d\sigma = 0, 1 \leq k \leq N.$$

Using (2.4) along with well known trace theorems and variational methods, we deduce the existence of a unique $u = u(\cdot, \epsilon) : \Omega \cap B(0, \rho) \rightarrow \mathbb{R}^N$ with $u^k \in W^{1,p}(\Omega \cap B(0, \rho)), 1 \leq k \leq N$, satisfying $\int_{\Omega \cap B(0, \rho)} u dx = 0$ and

$$(2.5) \quad \int_{\Omega \cap B(0, \rho)} (\epsilon + |\nabla u|^2)^{p/2-1} \langle \nabla u, \nabla v \rangle dx = \int_{\Omega \cap \partial B(0, \rho)} \langle f, v \rangle dx$$

whenever $v = (v^1, \dots, v^N)$ with $v^k \in W^{1,p}(\Omega \cap B(0, \rho))$. In (2.5) we have used v on $\Omega \cap \partial B(0, \rho)$ to denote the trace of v . More specifically, u is the minimum of the functional

$$F_\epsilon = \frac{1}{p} \int_{\Omega \cap B(0, \rho)} (\epsilon + |Dw|^2)^{p/2} dx - \int_{\partial B(0, \rho) \cap \Omega} \langle f, w \rangle$$

among all $w \in W^{1,p}$ such that $\int_{\Omega \cap B(0, \rho)} w = 0$. Existence follows from lower semicontinuity of F_ϵ , (2.4), and compactness of the trace operator. Choosing $v = u$ in (2.5) and using (2.4), Poincaré's inequality, (2.1), we deduce that

$$(2.6) \quad \int_{B(0, \rho)} (\epsilon + |\nabla u|^2)^{p/2-1} |\nabla u|^2 dx \leq c.$$

Next we claim that

$$(2.7) \quad u^k \in C^\infty(\overline{\Omega \cap B(0, 2)}) \text{ for } 1 \leq k \leq N \text{ and } 1 \leq m \leq n.$$

Constants in (2.7) however may depend on ϵ, p, n, N and the smoothness of $\partial\Omega$ in (2.3). We sketch the proof of (2.7) in the appendix to this paper for the reader's convenience since we have not been able to find a suitable reference. Our proof uses a reflection type argument, after straightening $\partial\Omega$ in an appropriate way, to first get $C^{1,\alpha}$ regularity in the closure of $\Omega \cap B(0, 2)$. After that we use Schauder type arguments to bootstrap. as in [ADN] , [ADN1].

From (2.5) and (2.7) we see that u is a strong solution to

$$(2.8) \quad \nabla \cdot \left((\epsilon + |\nabla u|^2)^{p/2-1} \nabla u \right) = 0$$

in $\Omega \cap B(0, \rho)$, where $\nabla \cdot$ denotes the divergence operator. Let $\lambda \in \mathbb{R}^n$ with $|\lambda| = 1$. Let $u_\lambda = (\nabla u_\lambda^1, \dots, \nabla u_\lambda^N)$ denote the directional derivative of u in the direction λ . Differentiating (2.8) with respect to λ we get for fixed $l, 1 \leq l \leq N$, that u_λ^m is a solution in $\Omega \cap B(0, \rho)$ to

$$(2.9) \quad \sum_{m=1}^N \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(b_{ij}^{lm} \frac{\partial}{\partial x_j} (u_\lambda^m) \right) = 0$$

where

$$(2.10) \quad b_{ij}^{lm}(x) = (\epsilon + |\nabla u|^2)^{p/2-2} \left((p-2) u_{x_j}^m u_{x_i}^l + \delta_{ij}^{lm} (\epsilon + |\nabla u|^2) \right) (x)$$

when $x \in \Omega \cap B(0, \rho)$. Here $\delta_{ij}^{lm} = 1$ if $i = j, l = m$, and is zero otherwise. We note that if $\xi = (\xi_k^\nu)$ is an $n \times N$ matrix, $|\xi|^2 = \langle \xi, \xi \rangle$ where the inner product is relative to \mathbb{R}^{nN} and $h = (\epsilon + |\nabla u|^2)^{1/2}$, then at $x \in \Omega \cap B(0, \rho)$

$$(2.11) \quad \min(p-1, 1) |\xi|^2 h^{p-2} \leq \sum_{m,l=1}^N \sum_{i,j=1}^n b_{ij}^{lm} \xi_i^l \xi_j^m \leq \max(p-1, 1) |\xi|^2 h^{p-2}.$$

For $1 \leq i \leq j \leq n$ let

$$(2.12) \quad c_{ij}(x) = (\epsilon + |\nabla u|^2(x))^{-1} \left[(p-2) \left(\sum_{\nu=1}^N u_{x_i}^\nu u_{x_j}^\nu \right) + \delta_{ij} (\epsilon + |\nabla u|^2) \right]$$

where δ_{ij} is the Kronecker δ and let L be the differential operator,

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(c_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

We observe that if $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, then

$$(2.13) \quad \min(p-1, 1) |\mu|^2 \leq \sum_{i,j=1}^n c_{ij}(x) \mu_i \mu_j \leq \max(p-1, 1) |\mu|^2.$$

We use (2.12), (2.13) to give a proof of a well known sub solution inequality for $L(h^q)$. To this end let e_k be the point in \mathbb{R}^n with a one in the k th

position and zeroes elsewhere. If $q \geq p$ at $x \in \Omega \cap B(0, \rho)$, we calculate

$$\begin{aligned}
 (2.14) \quad L(h^q) &= \sum_{i,j=1}^n (c_{ij}(h^q)_{x_j})_{x_i} \\
 &= q(q-p)h^{q-2} \sum_{i,j=1}^n c_{ij} h_{x_i} h_{x_j} + (q/p)h^{q-p}L(h^p).
 \end{aligned}$$

Moreover,

$$(2.15) \quad L(h^p) = p \sum_{i,j,k=1}^n \sum_{m=1}^N (h^{p-4} u_{x_j}^l u_{x_i}^l u_{x_k}^m u_{x_k x_j}^m)_{x_i} + \Delta(h^p) = S_1 + \Delta(h^p)$$

where Δ is the Laplacian. Thus

$$\begin{aligned}
 (2.16) \quad S_1 &= p(p-2) \sum_{i,j,k=1}^n \sum_{m,l=1}^N h^{p-4} u_{x_j x_i}^l u_{x_i}^l u_{x_k}^m u_{x_k x_j}^m \\
 &\quad + p(p-2) \sum_{i,j,k=1}^n \sum_{m,l=1}^N u_{x_j}^l (h^{p-4} u_{x_i}^l u_{x_k}^m u_{x_k x_j}^m)_{x_i} = S_3 + S_4
 \end{aligned}$$

Using (2.9) with k playing the role of j and x_j the role of λ we have

$$(2.17) \quad S_4 = -p \sum_{i=1}^n \sum_{l=1}^N u_{x_j}^l (h^{p-2} u_{x_j x_i}^l)_{x_i} = -\Delta(h^p) + p \sum_{i,j=1}^n \sum_{l=1}^N h^{p-2} (u_{x_j x_i}^l)^2.$$

Combining (2.15)-(2.17) it follows that $L(h^p) \geq p \min(1, p-1) h^{p-2} |\nabla h|^2$ and thereupon from (2.13), (2.14) that for some $c = c(p) \geq 1$,

$$(2.18) \quad L(h^q) \geq c^{-1} q^2 h^{q-2} |\nabla h|^2 = (4c)^{-1} |\nabla(h^{q/2})|^2.$$

Let $\phi \in C_0^\infty(B(0, 3/2))$ and put $w = \phi^2$. From (2.18), (2.3), (2.7), and integration by parts we find for $q \geq p$ a constant $c = c(p, n) \geq 1$ with

$$\begin{aligned}
 (2.19) \quad T &= c^{-1} \int_{\Omega \cap B(0,4)} \phi^2 |\nabla h^{q/2}|^2 dx \leq \int_{\Omega \cap B(0,4)} \phi^2 L(h^q) dx \\
 &= \sum_{i,j=1}^n \int_{\partial\Omega} \phi^2 c_{ij}(h^q)_{x_j} \nu_i d\sigma - 2 \sum_{i,j=1}^n \int_{\Omega \cap B(0,4)} \phi \phi_{x_i} c_{ij}(h^q)_{x_j} dx = T_1 + T_2.
 \end{aligned}$$

From (2.5), smoothness of $\partial\Omega$ and u , we deduce that $u_\nu = 0$ on $\partial\Omega \cap B(0, 2)$. Using this deduction and (2.12), we see that only the δ_{ij} term contributes to the sum defining T_1 . This remark and [G](pp. 133-137 and Equation 3.1.1.8) yield

$$\begin{aligned}
(2.20) \quad T_1 &= q \int_{\partial\Omega} \phi^2 h^{q-2} \sum_{j,k=1}^n \sum_{l=1}^N u_{x_k}^l \nu_j u_{x_k x_j}^l d\sigma \\
&= \int_{\partial\Omega} \sum_{l=1}^N \phi^2 h^{q-2} M(\nabla_t u^l, \nabla_t u^l) d\sigma \leq 0
\end{aligned}$$

where $M(.,.)$ is the second fundamental quadratic form on $\partial\Omega$ and $\nabla_t u$ is the tangential component of ∇u on $\partial\Omega$. Since Ω is convex, $M(.,.)$ is nonpositive. Thus $T_1 \leq 0$. Also using Cauchy's inequality with δ 's we get

$$(2.21) \quad |T_2| \leq \delta \int \phi^2 |\nabla(h^{q/2})|^2 dx + \frac{c}{\delta} \int |\nabla\phi|^2 h^q dx$$

Choosing δ so small that the first term on the righthand side is $\leq T/2$ and then using (2.21), (2.20), in (2.19) we conclude after some arithmetic that

$$(2.22) \quad \int \phi^2 |\nabla h^{q/2}|^2 dx \leq c \int |\nabla\phi|^2 h^q dx$$

where $c = c(p, n) \geq 1$. From (2.22) and Sobolev's inequality for $\Omega \cap B(0, 2)$ applied to $\theta = h^{q/2}\phi$ we deduce for $n \neq 2$ that

$$\begin{aligned}
(2.23) \quad & \|\theta\|_{L^{2n/(n-2)}(\Omega \cap B(0,2))}^2 \leq (c/|\Omega \cap B(0,2)|^2) \|\nabla\theta\|_{L^2(\Omega \cap B(0,2))}^2 \\
& \leq (c/|\Omega \cap B(0,2)|^2) \int_{\Omega \cap B(0,2)} |\nabla\phi|^2 h^q dx
\end{aligned}$$

If $n = 2$ replace $2n/(n-2)$ by 4 in (2.23). We can now use Moser iteration (see[GT, ch 8]) in a well known way and get for every $x \in \Omega \cap B(0, 1)$ that

$$(2.24) \quad (\epsilon + |\nabla u|^2)^{p/2}(x) \leq C \int_{\Omega \cap B(0,2)} (\epsilon + |\nabla u|^2)^{p/2} dx$$

where C has the same dependence as in Theorem 1.1. Finally, we note (see [D], [L], [T] for $N = 1$ and [T1] for $N > 1$) that $\nabla u(\cdot, \epsilon)$ is Hölder continuous on compact subsets of $\tilde{\Omega} \cap B(0, \rho)$ with Hölder constants independent of ϵ . Also from (2.6) we find that $\{u(\cdot, \epsilon)\}$ is uniformly bounded in $W^{1,p}(\tilde{\Omega} \cap B(0, \rho))$. Using these facts it follows easily that subsequences of $\{u(\cdot, \epsilon)\}, \{\nabla u(\cdot, \epsilon)\}$ converge uniformly on compact subsets of $\Omega \cap B(0, \rho)$ as $\epsilon \rightarrow 0$ to $u' \in W^{1,p}(\tilde{\Omega} \cap B(0, \rho))$. From uniform integrability type arguments we see that (2.5) holds for $u', \tilde{\Omega} \cap B(0, \rho)$ with $\epsilon = 0$ when v is infinitely differentiable on \mathbb{R}^n . Since a given $v \in W^{1,p}(\Omega \cap B(0, \rho))$ can be approximated arbitrarily closely in the norm of this space by such functions, we conclude that (2.5) is valid with $\epsilon = 0$ and u, Ω replaced by $u', \tilde{\Omega}$. Now \tilde{u} is the unique function (up to a constant) having these properties so $u' - \tilde{u} = \text{constant}$. Thus (2.6), (2.24) hold with $\epsilon = 0$ when u, Ω are replaced by $\tilde{u}, \tilde{\Omega}$. Hence Theorem 1.1 is true. \square

3. APPENDIX: PROOF OF 2.7

After a rotation if necessary, around a neighborhood of any boundary point $y_0 \in \partial\Omega \cap B(0, 2)$, $\partial\Omega$ can be locally represented as $\{y', y_n) : y_n = \phi(y')\}$ and $\Omega = \{(y', y_n) : y_n > \phi(y')\}$ for some $\phi \in C^\infty(R^{n-1})$. As in [GT] (Section 14.6), it follows that there exists $\mu > 0$ such that if $x \in \Omega$ and $d(x, \partial\Omega) = d(x) < \mu$, then there is a unique point $y(x) \in \partial\Omega$, such that $|y - x| = d(x, \partial\Omega) = d(x)$. The points x and y are related by

$$(3.1) \quad x = y - \nu(y)d$$

where ν is the outer unit normal to Ω

As proved in Lemma 14.16 in [GT], the map $g(y', d) = (y', \phi(y')) - \nu(y', \phi(y'))d$ is locally invertible in a neighborhood U of $(y'_0, 0)$ and maps $U \cap \{d > 0\}$ into Ω . Hence by the inverse function theorem y and d in (3.1) are locally C^∞ functions of x . Using $\nabla d(x) = -\nu(y)$ one now calculates that the inverse of g is the mapping,

$$(3.2) \quad g^{-1}(x) = (y', d) \text{ where } d = d(x) \text{ and } y_i = x_i - d(x)d_{x_i}(x), 1 \leq i \leq n-1.$$

By shrinking U if necessary, we may assume that $g(U \cap \{d > 0\})$ is contained in $\Omega \cap B(0, 5/2)$.

We note from (3.2) that at $x \in g(U)$,

$$(3.3) \quad \langle \nabla y_i, \nabla d \rangle = \langle e_i - (\nabla d_{x_i})d - d_{x_i} \nabla d, \nabla d \rangle = -d \langle \nabla d_{x_i}, \nabla d \rangle = 0$$

where in the last inequality we have used the fact that ∇d is constant along the line from x to y in (3.1) (so the directional derivative of d_{x_i} in the direction of $\nabla d(x) = 0$).

Define $\hat{u}(y', d) = u(x)$ where u is as in (2.5). We have

$$(3.4) \quad \langle \nabla u, \nabla v \rangle = \left\langle \sum_{i=1}^{n-1} \hat{u}_{y_i} \nabla y_i + \hat{u}_d \nabla d, \sum_{i=1}^{n-1} \hat{v}_{y_i} \nabla y_i + \hat{v}_d \nabla d \right\rangle$$

Using the orthogonality of ∇y_i and ∇d , we get

$$(3.5) \quad \langle \nabla u, \nabla v \rangle = \hat{u}_d \hat{v}_d + \sum_{i,j=1}^{n-1} \hat{u}_{y_i} \hat{v}_{y_j} q_{ij}(y', d)$$

In conclusion, by renaming the variable $d = y_n$ and using the change of variables formula, we deduce from (2.8), (3.5) that \hat{u} satisfies

$$(3.6) \quad \int_{U \cap y_n > 0} (\epsilon + \langle A(y) \nabla \hat{u}, \nabla \hat{u} \rangle)^{p/2-1} \langle A(y) \nabla \hat{u}, \nabla \zeta \rangle V(y', y_n) dy = 0$$

where U is as above and ζ is in $W_0^{1,p}(U)$. V is the Jacobian of g . The matrix $A = (a_{ij})$ is defined by $a_{ij} = q_{ij}$ when $1 \leq i, j \leq n-1$, where q_{ij} is as

in (3.5), $a_{kn} = a_{nk} = 0$ when $k < n$ and $a_{nn} = 1$. Also if $z \in \mathbb{R}^n$, then $A(y)z = \eta$ where $\eta = (\eta_1, \dots, \eta_n)$ with

$$\eta_i = \sum_{j=1}^n a_{ij}(y)z_j \text{ and } A\nabla\hat{u} = (A\nabla\hat{u}^1, \dots, A\nabla\hat{u}^N).$$

Now let $\xi = \zeta V$. Using ξ in place of ζ in (3.6), we obtain

$$(3.7) \quad \int_{U \cap \{y_n > 0\}} (\epsilon + \langle A(y)\nabla\hat{u}, \nabla\hat{u} \rangle)^{p/2-1} \langle A(y)\nabla\hat{u}, \nabla\xi \rangle dy$$

$$= \int_{U \cap \{y_n > 0\}} (\epsilon + \langle A(y)\nabla\hat{u}, \nabla\hat{u} \rangle)^{p/2-1} \sum_{l=1}^N \langle A(y)\nabla\hat{u}^l, \nabla V \rangle \xi^l V^{-1} dy$$

Now we extend A , \hat{u} and V to $U \cap \{y_n < 0\}$ by even reflection and denote them by B , \hat{v} and W respectively. That is, $B(y', y_n) = A(y', -y_n)$, $\hat{v}(y', y_n) = \hat{u}(y', -y_n)$, and $W(y', y_n) = V(y', -y_n)$ for $y_n < 0$. We note that B and W are Lipschitz extensions of A and V respectively and $\hat{v} \in W^{1,p}(U)$

Now for any $\zeta \in W_0^{1,p}(U)$

$$(3.8) \quad \int_U (\epsilon + \langle B(y)\nabla\hat{v}, \nabla\hat{v} \rangle)^{p/2-1} \langle B(y)\nabla\hat{v}, \nabla\zeta \rangle dy = \int_{U \cap \{y_n > 0\}} + \int_{U \cap \{y_n < 0\}}.$$

For the first integral on the right hand side of (3.8), we use (3.7) and the fact that \hat{v} , W restricted to $\{y_n > 0\}$ equal \hat{u} , V respectively. For the second integral, we define $\psi(y', -y_n) = \zeta(y', y_n)$ and change variables. Using the definition of \hat{v} , ψ and W we obtain (3.7) with ξ replaced by ψ . Altogether we see that

$$(3.9) \quad \int_U (\epsilon + \langle B(y)\nabla\hat{v}, \nabla\hat{v} \rangle)^{p/2-1} \langle B(y)\nabla\hat{v}, \nabla\zeta \rangle dy$$

$$= \int_U (\epsilon + \langle B(y)\nabla\hat{v}, \nabla\hat{v} \rangle)^{p/2-1} \sum_{l=1}^N \langle B(y)\nabla\hat{v}^l, \nabla W \rangle \zeta^l W^{-1} dy$$

Thus $\hat{v} \in W^{1,p}(U)$ is a weak solution in U to a system of the form

$$(3.10) \quad \nabla \cdot \hat{b}(y, \nabla\hat{v}) - \hat{b}_0(y, \nabla\hat{v}) = 0.$$

Moreover this system satisfies the structural assumption in [T1] (see (1.1) and (1.7)-(1.13) in this paper) except that in [T1], the analogue of \hat{b} is assumed to be C^1 in y . However estimates in [T1] only use Lipschitz norms in y so are also valid in our case. To check the lower order term observe that the l th component of $\hat{b}_0(y, \nabla\hat{v})$, $1 \leq l \leq N$, is given by

$$(3.11) \quad (\epsilon + \langle B(y)\nabla\hat{v}, \nabla\hat{v} \rangle)^{p/2-1} \langle B(y)\nabla\hat{v}^l, \nabla W \rangle W^{-1}.$$

This term may be discontinuous in y when $y_n = 0$ thanks to W_{y_n} , but still satisfies the growth and structure assumptions in (1.13) of [T1]. Therefore, we conclude from [T1], that $\hat{v} \in C_{loc}^{1,\alpha}(U) \cap W_{loc}^{2,2}(U)$ (for second derivative estimates, see section 4 in [T1]).

We note from the above results and (3.10), (3.11), that $\hat{b}_0(y, \nabla \hat{v}(y))$ is Hölder continuous in U since $\hat{v}_{y_n} = 0$ at points in U where $y_n = 0$. Moreover $(b_{ij})_{y_k}$ are Lipschitz continuous when $k < n$. Therefore, by using a difference quotient argument we deduce that if $\lambda = y_k$ for $k < n$ then for $1 \leq m \leq N$, (3.10) can be differentiated with respect to λ as in the derivation of (2.9) in order to obtain that $\hat{v}_\lambda^m = \langle \nabla \hat{v}^m, \lambda \rangle$, $1 \leq m \leq N$, is a weak solution to a uniformly elliptic linear system in N equations of the form,

$$(3.12) \quad \sum_{m=1}^N \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(C_{ij}^{ml}(y) (\hat{v}_\lambda^m)_{y_j} \right) = \sum_{i=1}^n (f_i^l)_{y_i}(y)$$

for fixed l , where f_i^l, C_{ij}^{ml} , $1 \leq i, j \leq n, 1 \leq l, m \leq N$, are Hölder continuous. We can now apply Theorem 2.2 in Chapter 3 of [Gi] to conclude that \hat{v}_λ has Hölder continuous derivatives in U , i.e $\hat{v}_{y_i y_j}$ is Hölder continuous in U when $i + j < 2n$. Now coupled with the fact that \hat{v} is in $W^{2,2}(U)$ we can write (3.10) in non-divergence form and obtain for each $l = 1, \dots, N$ that

$$(3.13) \quad \sum_{m=1}^N a^{lm} \hat{u}_{y_n y_n}^m = h^l \text{ at } y \in U \text{ with } y_n > 0$$

where h^l is Hölder continuous in the closure of $U \cap \{y : y_n > 0\}$ and

$$a^{lm} = \delta_{lm} + (p-2) \frac{\hat{u}_{y_n}^l \hat{u}_{y_n}^m}{(\epsilon + \langle A(x) \nabla \hat{u}, \nabla \hat{u} \rangle)}$$

Thus we see that (a^{lm}) as a matrix is Hölder continuous and positive definite at each point. Consequently, the linear equations corresponding to (3.13) can be solved and $\hat{u}_{y_n y_n}^l$ is expressible in terms of functions which are Hölder continuous in the closure of $U \cap \{y_n > 0\}$. Therefore $\hat{u} \in C^{2,\alpha}$ upto $y_n = 0$. Using the definition of \hat{u} , we then obtain that u is $C^{2,\alpha}$ in the closure of $\Omega \cap g(U)$. Interior estimates are similar so we conclude that u is $C^{2,\alpha}$ for some $\alpha > 0$ in the closure of $\Omega \cap B(0, 2)$.

We can now use Schauder type arguments as in [ADN] for equations and [ADN1] for systems (see also [Li] for quasilinear equations) to bootstrap and eventually deduce that u is infinitely differentiable in the closure of $\Omega \cap B(0, 2)$. The proof of (2.7) is now complete.

Remark: We emphasize that in order to satisfy the hypotheses in [T1] (i.e to get an even Lipschitz extension of A), it was important that $a_{kn} = 0$ for

$k < n$. This is precisely why we chose our coordinates using the distance function.

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